SECTION 8.3: TRIGONOMETRIC INTEGRALS

INTEGRALS INVOLVING POWERS OF SINE AND COSINE:

RECALL: For all real numbers, t, $\cos^2(t)+\sin^2(t)=1$. This lets us 'exchange' sines and cosines as follows:

- To convert sines into cosines: $\sin^2(t)=1-\cos^2(t)$; moreover, $\sin^{2k}(t)=\left(\sin^2(t)\right)^k=\left(1-\cos^2(t)\right)^k$
- To convert cosines into sines: $\cos^2(t)=1-\sin^2(t)$; moreover, $\cos^{2k}(t)=\left(\cos^2(t)\right)^k=\left(1-\sin^2(t)\right)^k$
- The so-called 'power reduction' formulas (a.k.a. 'double angle' formulas):

$$\sin^2(t)=\frac{1-\cos(2t)}{2};\quad \text{moreover,}\quad \sin^{2k}(t)=\left(\sin^2(t)\right)^k=\left(\frac{1-\cos(2t)}{2}\right)^k$$

$$\cos^2(t)=\frac{1+\cos(2t)}{2};\quad \text{moreover,}\quad \cos^{2k}(t)=\left(\cos^2(t)\right)^k=\left(\frac{1+\cos(2t)}{2}\right)^k$$

- If $u=\sin(t)$, then $du=\cos(t)\,dt$ and if $u=\cos(t)$, then $du=-\sin(t)\,dt$.

The above observations lead to the following strategies:

STRATEGIES: Given an integral involving powers of sine and cosine ...

- Can you spare (factor out) a sine and convert the leftovers into cosines? If so, let $u=$ cosine.
- Can you spare (factor out) a cosine and convert the leftovers into sines? If so, let $u=$ sine.
- If neither of the above work, try the 'power reduction' formulas.

EXAMPLE 1: Find the following antiderivatives (indefinite integrals). Check your answers.

1. $\displaystyle\int \cos^3(t)\,dt$ — Ans: $\sin(t)-\frac{1}{3}\sin^3(t)+C$

2. $\displaystyle\int \frac{\sin^3(2\theta)}{\cos(2\theta)}\,d\theta$ — Ans: $-\frac{1}{2}\ln|\cos(2\theta)|+\frac{1}{4}\cos^2(2\theta)+C$

3. $\displaystyle\int_0^{\frac{2\pi}{3}} \sin^4(3x)\,dx$ — Ans: $\frac{\pi}{4}$

EXAMPLE 2: (VIDEO) Find $\displaystyle\int \sin^3(t)\cos(t)\,dt$ using two different substitutions.

Show your answers are equivalent. Ans: $\frac{1}{4}\sin^4(t)+C$ or $\frac{1}{4}\cos^4(x)-\frac{1}{2}\cos^2(x)+C$

EXAMPLE 3: (VIDEO) Find $\displaystyle\int_0^{4\pi}\sqrt{1-\cos(x)}\,dx$. Ans: $8\sqrt{2}$.

HINT: $\sin^2(t)=\dfrac{1-\cos(2t)}{2}$ so $1-\cos(2t)=2\sin^2(t)$ ...

INTEGRALS INVOLVING POWERS OF (CO)SECANTS AND (CO)TANGENTS:

RECALL: For all real numbers, t, $1+\tan^2(t) = \sec^2(t)$. This lets us 'exchange' secants and tangents as follows:

- To convert secants into tangents: $\sec^2(t) = 1+\tan^2(t)$; moreover, $\sec^{2k}(t) = \left(\sec^2(t)\right)^k = \left(1+\tan^2(t)\right)^k$
- To convert tangents into secants: $\tan^2(t) = \sec^2(t)-1$; moreover, $\tan^{2k}(t) = \left(\tan^2(t)\right)^k = \left(\sec^2(t)-1\right)^k$
- If $u = \sec(t)$, then $du = \sec(t)\,\tan(t)\,dt$ and if $u = \tan(t)$, then $du = \sec^2(t)\,dt$.

The above observations lead to the following strategies:

STRATEGIES: Given an integral involving powers of (co)secants and (co)tangents ...

- Can you spare (factor out) a $(\text{secant})^2$ and convert the leftovers into tangents? If so, let $u =$ tangent.
- Can you spare (factor out) a (secant)(tangent) convert the leftovers into secant? If so, let $u =$ secant.
- If you have no secants, make some: $(\text{tangent})^2 = (\text{secant})^2 - 1$.
- If you have an odd power of secants and no tangents, use parts with 'dv' being $(\text{secant})^2$.

NOTE: For integrals involving cosecants and cotangents, use the appropriate 'co'-strategies!

EXAMPLE 4: Find the following antiderivatives (indefinite integrals). Check your answers.

1. $\displaystyle\int \sec^6(2x)\,dx$ Ans: $\displaystyle\frac{1}{2}\tan(2x) + \frac{1}{3}\tan^3(2x) + \frac{1}{10}\tan^5(2x) + C$

2. $\displaystyle\int \sec^{\frac{3}{2}}(t)\tan^3(t)\,dt$ Ans: $\displaystyle\frac{2}{7}\sec^{\frac{7}{2}}(t) - \frac{2}{3}\sec^{\frac{3}{2}}(t) + C$

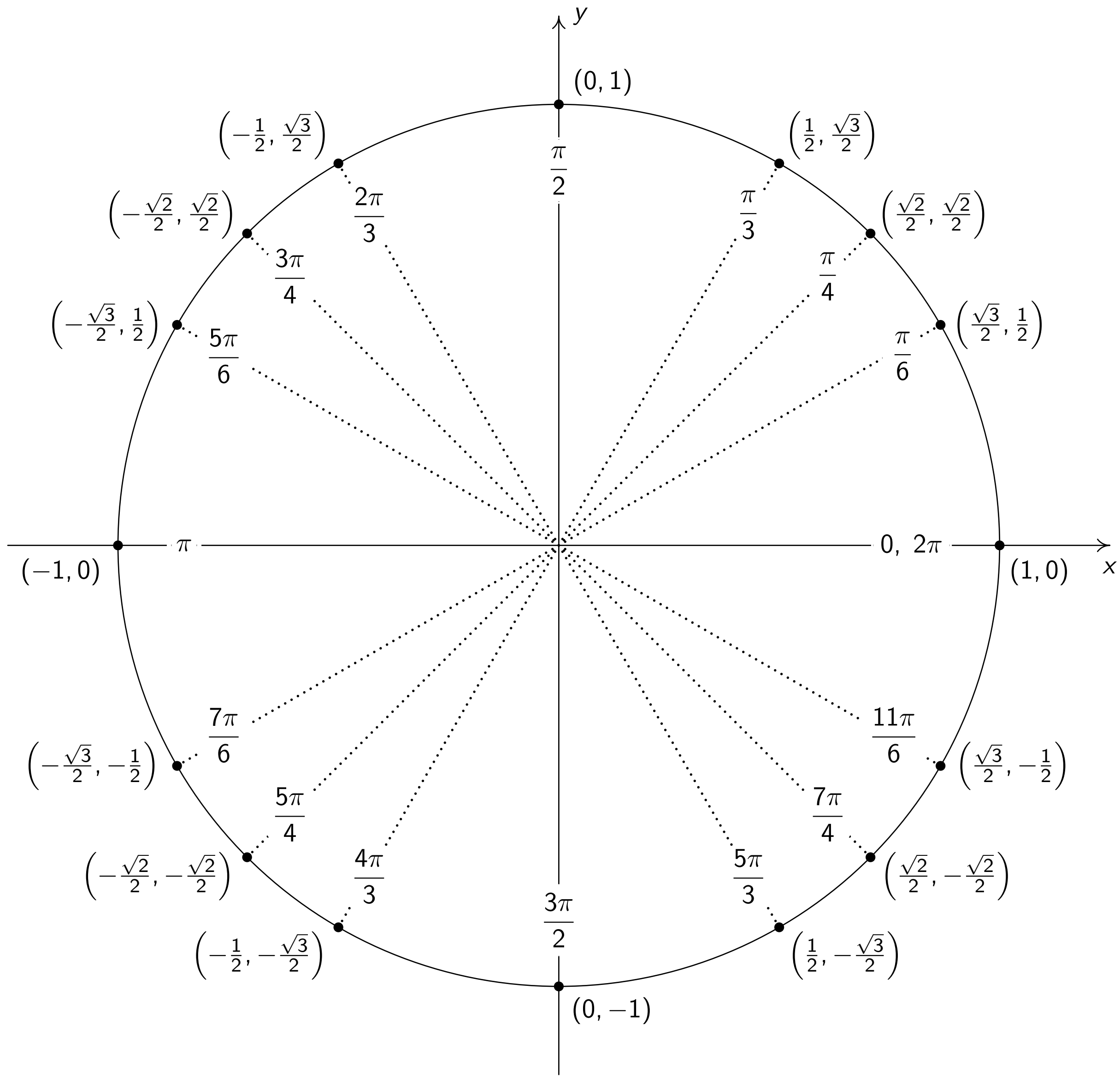
3. $\displaystyle\int \tan^3(3x)\,dx$ Ans: $\displaystyle\frac{1}{6}\tan^2(3x) - \frac{1}{3}\ln|\sec(3x)| + C$

EXAMPLE 5: (VIDEO) Find $\displaystyle\int \csc^3(x)\,dx$. Ans: $\displaystyle\frac{-\csc(x)\cot(x) + \ln|\csc(x) - \cot(x)|}{2} + C$

HINT: You *could* use parts with $dv = \csc^2(x)\,dx$ **OR** you could use our formula for $\displaystyle\int \sec^3(x)\,dx$ ...

HOMEWORK: Section 8.3: 9 - 61 odd, 75*

Math 1700 Important Points on the Unit Circle



Math 1700 Identities

- Reciprocal and Quotient Identities: $$\begin{cases} \tan(\theta) &= \dfrac{\sin(\theta)}{\cos(\theta)} \\ \cot(\theta) &= \dfrac{\cos(\theta)}{\sin(\theta)} \\ \csc(\theta) &= \dfrac{1}{\sin(\theta)} \\ \sec(\theta) &= \dfrac{1}{\cos(\theta)} \end{cases}$$

- Pythagorean:

  Relating sine and cosine: $$\begin{cases} \sin^2(\theta)+\cos^2(\theta) &= 1 \\ \sin^2(\theta) &= 1-\cos^2(\theta) \\ \cos^2(\theta) &= 1-\sin^2(\theta) \end{cases}$$

  Relating secant and tangent: $$\begin{cases} 1+\tan^2(\theta) &= \sec^2(\theta) \\ \tan^2(\theta) &= \sec^2(\theta)-1 \\ 1 &= \sec^2(\theta)-\tan^2(\theta) \end{cases}$$

  Relating cosecant and cotangent: $$\begin{cases} 1+\cot^2(\theta) &= \csc^2(\theta) \\ \cot^2(\theta) &= \csc^2(\theta)-1 \\ 1 &= \csc^2(\theta)-\cot^2(\theta) \end{cases}$$

- Even/Odd Identities: $$\begin{cases} \sin(-\theta) &= -\sin(\theta) \\ \cos(-\theta) &= \cos(\theta) \\ \tan(-\theta) &= -\tan(\theta) \end{cases}$$

- Sum/Difference Identities: $$\begin{cases} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \\ \cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) \\ \tan(\alpha+\beta) &= \dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \\ \tan(\alpha-\beta) &= \dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)} \end{cases}$$

- Double Angle Identities: $\begin{cases} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta) \\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ \tan(2\theta) &= \dfrac{2\tan(\theta)}{1-\tan^2(\theta)} \end{cases}$

- Double Angle Identities for use in Calculus: $\begin{cases} \cos^2(\theta) &= \dfrac{1+\cos(2\theta)}{2} \\ \sin^2(\theta) &= \dfrac{1-\cos(2\theta)}{2} \end{cases}$

- Half Angle Identities: $\begin{cases} \cos\left(\dfrac{\theta}{2}\right) &= \pm\sqrt{\dfrac{1+\cos(\theta)}{2}} \\ \sin\left(\dfrac{\theta}{2}\right) &= \pm\sqrt{\dfrac{1-\cos(\theta)}{2}} \\ \tan\left(\dfrac{\theta}{2}\right) &= \dfrac{1-\cos(\theta)}{\sin(\theta)} = \dfrac{\sin(\theta)}{1+\cos(\theta)} \end{cases}$

- Product to Sum Identities: $\begin{cases} \cos(\alpha)\cos(\beta) &= \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right) \\ \sin(\alpha)\sin(\beta) &= \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right) \\ \sin(\alpha)\cos(\beta) &= \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right) \end{cases}$

- Sum to Product Identities: $\begin{cases} \cos(\alpha)+\cos(\beta) &= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) \\ \cos(\alpha)-\cos(\beta) &= -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right) \\ \sin(\alpha)+\sin(\beta) &= 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) \\ \sin(\alpha)-\sin(\beta) &= 2\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right) \end{cases}$